# CHAPTER 23

# Improving Understanding

A course is more than a collection of definitions, theorems and proofs. If the teacher has done their job correctly, then there is a common theme and a well-defined stock of techniques and ideas. For true understanding of mathematics one should look for the theme and for the repeatedly used techniques.

Complex analysis is no exception. The theme I have chosen in this book is to generalize the notions of calculus to the complex numbers in order to solve problems involving real numbers. The chapters on real integrals are good examples of this. The statements of the problems are purely real – e.g., sum a real series – while the solution is found by going through the complex domain. It is possible to create a course that omits these applications and generalizes for the sake of generalization and the motivation for study comes from the elegance of the resulting theory.

The main technique used here is the Estimation Lemma. However, the main *theoretical* result upon which I build results is Cauchy's Theorem. From it we prove Cauchy's Integral Formula, Taylor's Theorem and Cauchy's Residue Theorem. In each of these the Estimation Lemma was used but it is Cauchy's Theorem that makes Complex Analysis what it is.

In any course, just as important as to what to include is what to exclude, or at least not emphasize as much. One area I have underplayed is topology. I could have included some more point set topology theorems. These could have been used in the proof of the Identity Theorem for example. Instead I opted to build into proofs any required topology results. Another topological theorem is the Paving Lemma. This is considered central in the classic textbook *Complex Analysis* by Stewart and Tall but here it is relegated to Appendix A to help prove the general form of Cauchy's Theorem.

## Basics of improving understanding

Higher level mathematics involves a stronger emphasis on statements and proofs rather than simple techniques such as finding the roots of a quadratic. In Complex Analysis we have a good blend of theory and simple techniques. For example, Cauchy's Residue Theorem relies on some serious theoretical results but its application involves finding winding numbers and residues, both of these usually involves simple calculations.

For the theoretical results it is very important to state the theorems with care. As described in the Common Mistakes chapter we do not state a theorem by just giving the conclusion, for example stating Cauchy's Integral Formula theorem as only

$$\int_\gamma \frac{f(z)}{z-w}\,dz = 2\pi i\, n(\gamma, w) f(w).$$

It is crucial that the assumptions are included and for deeper understanding you need to know *why* they are included. If an assumption is dropped or changed, then what difference does it have on the result?

The assumptions can be crucial in subtle ways. This can be seen in Theorem 8.9, The Fundamental Theorem of Calculus. One of the conditions is that there exists $F$ such that $F' = f$. It does not say that there exists one. Not realizing this and internalizing it leads to the common misconception that one could directly prove Cauchy's Theorem from it. (The fallacious argument is that $\int f(z)\,dz = F(\gamma(b)) - F(\gamma(b))$ and since the contour is closed, $\gamma(a) = \gamma(b)$ and so $\int f(z)\,dz = 0$.) Note here the use of the word 'directly'. This is again subtle. The Fundamental Theorem of Calculus *is* used in the proof of the simplified version of Cauchy's Theorem given earlier and in the full version in the Appendix! The point is that it is merely a small part of each proof.

One can get a further understanding of statements by consulting other books as this will often give insight into the essential part of a theorem. For example, in Stewart and Tall (see Further Reading below), Cauchy's Integral Formula is only stated for a circular contour traversed once. Priestley's statement (again, see Further Reading below) is only for simple closed positively oriented contours that are traversed once. If you look up her statement, then be aware that, unlike in this book, Priestley defines contours to be simple and closed.

## Some common patterns

The power of the Estimation Lemma has already been remarked upon. What other ideas are used repeatedly? One is the paradigm introduced in Remark 2.15. That is we prove results in complex analysis from results in real analysis. This is achieved by taking the modulus and the argument (or just one of them), or by taking real and imaginary parts. In fact, this can be seen throughout the book since the Estimation Lemma is a central example of this idea.

It can also be seen in results such as proving that complex power series are differentiable and differentiable term-by-term, the ratio and comparison tests used there are both real analysis results.

A lot of results rely on taking the modulus so here's a quick useful tip. Instead of using $|z|$, use $|z|^2$. This eliminates the square root so reduces clutter in the calculations and also we can clearly use $z\overline{z}$ as a possible substitution to help solve the problem.

## The Estimation Lemma

As stated earlier, the Estimation Lemma appears regularly and so to gain deeper understanding of Complex Analysis through learning and understanding proofs we should ask 'Where does the Estimation Lemma appear?' The proof can then be broken into parts: 1) getting to the Estimation Lemma, 2) the consequence of using the lemma. The next section deals with this more generally.

## Memorizing proofs

Reproducing proofs is important for exams and many students attempt to memorize proofs word-for-word. This is very inefficient. Proofs are not like lines in a play that need to be spoken precisely as given. In a particular proof there is usually some structure that we need to know and we can just improvise the rest around it. Learning proofs this way leads to deeper understanding of the material and moreover develops mathematical abilities.

In the previous section I claimed that finding where the Estimation Lemma is used reveals the structure of many proofs in Complex Analysis. If you haven't already done so, go through and identify proofs which use it. (You should see most do!) Thus when memorizing proofs in Complex Analysis one should aim to memorize where in the proof the Estimation Lemma is used. To apply the

Estimation Lemma we need a function and a contour. Those are the particulars we have to be memorize.

Let us take the proof of Liouville's Theorem as an example. The key point is to apply the Estimation Lemma to the integral in Taylor's Theorem that allows us to calculate the first derivative of the function. The contour is just a circle at $z_0$ and like in many other proofs and examples we let the radius go to infinity.

I would remember the proof simply as 'Use the Estimation Lemma on the integral from Taylor's Theorem to find the first derivative at a general point'.

Note that I don't write down extras such as 'Show $f'(z_0) = 0$ by letting the radius tend to infinity'. The reason for this is that most of the proofs and examples have that the radius goes to infinity or to zero. After all, what else are we going to do with it? It should be obvious which we should in each application. Here letting $r$ go to zero would make the bound go to infinity which is useless. Thus we let $r$ go to infinity. From there it is obvious that $f'(z_0) = 0$. It should then be equally obvious that this means $f$ is constant since $z_0$ was general.

To recap, we memorize this proof in one line. For other proofs we may have to use more steps. For example, in the proof of the Fundamental Theorem of Algebra, I would use

(i) Proof by contradiction.

(ii) Show bound $\left|\dfrac{p(z)}{z^n}\right| \geq \dfrac{|a_n|}{2}$ for large $|z|$.

(iii) Apply Estimation Lemma to Cauchy's Integral Formula for calculating $1/p(0)$.

(iv) Show $1/p(0)$ is 0 for the contradiction.

The proof is then as follows: From step (i) I know that my polynomial $p$ has no roots. Step (ii) reminds me to define $a_j$ in the polynomial and to work out how $\left|\dfrac{p(z)}{z^n}\right|$ behaves for large $|z|$.

Note that if I do not succeed in proving the correct bound for some reason (such as exam nerves), then by memorizing the bound I need, I can at least continue the proof from this point.

Next we apply the Estimation Lemma to an integral arising from Cauchy's Integral Formula. This gives $|1/p(0)| = 0$ which is impossible and so we have the required contradiction.

## Knowing facts versus looking them up

It is currently popular to claim that students should not learn anything they could look up. Inspirational quotes by Einstein such as ‘Education is not the learning of facts, but the training of the mind to think’ and ‘Never memorize something you can look up’ are shared on social media. Leaving aside whether he said these or not, the sharer’s point is fairly clear, memorization of facts has no place in a respectable education.

It is equally clear where this idea comes from. In the past much education was about reciting facts – I certainly spent a lot of my early mathematics lessons chanting times tables along with my classmates. Such memorization without understanding is dull and frustrating for the learner and it is easy to have sympathy with the ‘look it up approach’. Too much memorization of facts and too little understanding is uninteresting.

Despite this, I’m going to make a case for knowing facts that could be looked up.

A student came to my office with a complex analysis problem he was struggling with. My reaction in this type of situation is not to explain the answer on the board but to hand the student the pen and make them write whilst I ask probing questions in my, admittedly poor, impersonation of Socrates. My reasoning here is that I know that if I only explain, the student will nod, say yes to anything I say and will write down the answer in the hope of understanding it later. Having the pen and writing forces them to think and clarifies for me where their misunderstanding lies. Yes, it is more painful for them (and me) but they learn a lot more.

This particular student was progressing well in the solution until we got stuck because he needed the sine addition formula and didn’t know it by heart. (The formulae is $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and a method for deducing it is given in the next section.) He claimed that he didn’t need to know it as it was something he could look up when needed. Well, clearly, this wasn’t true. He needed it there and then and wasn’t able to look it up. Interestingly, and bafflingly, he repeatedly refused to let me tell him how to quickly deduce it. In the end he left my office promising that he would be able to sort out the problem.

Why am I concerned about this? Well, we had been diverted by a triviality. It is akin to halting in an arithmetical calculation to look up $7 + 5$. This minor problem had broken our concentration and focus on the important question. When learning a foreign language – and mathematics is very much a language – one could not achieve fluency if one had to always look up vocabulary or how to conjugate a verb. Here lack of knowing impacted fluency. Knowing enables fluency.

But let's also look at the wider picture. What is learning? How can we define it? One definition is 'the transfer of facts and skills into long term memory'. You really know something when you can instantly remember it. For example, you can claim to have learned the basics of driving when the actions involved are pushed down from the conscious to the subconscious. No one can claim to be able to drive if, every time, they had to consciously think through how to overtake or how to change gear.

Another important part of learning is to make connections between topics and facts. We often feel we have gained a deep insight when we see how two seemingly different areas are connected. For example, viewing complex numbers as vectors in the plane. But how can you make a connection between facts if you don't know those facts?

So there are some serious reasons to memorize facts and procedures. In particular to not waste time, improve fluency and to make connections between areas.

That is not to say one should memorize every fact from every course taken and I should make clear that I am not in favour of mindless rote learning. The memorization is there to aid problem-solving and understanding and is definitely not the goal of education.

My favoured approach is to abandon the choice between the "memorization" and "look-it-up" mindsets. The key skill is to identify what should be memorized and what should be looked up. For any particular fact it becomes a matter of choice and depends on what one uses regularly. Something used day-to-day will likely become memorized anyway. It is the material that is used every so often where a decision needs to be made.

My belief is that the sine formula should be memorized as it is used in different areas – analysis, geometry, and ordinary differential equations to name a small selection. Whether something like the Cauchy-Riemman equations will need to be memorized will depend on the course of study. Obviously it should be learned by heart for an exam as generally one can't look up in an exam and besides it would waste time working it out. (A method to remember the Cauchy-Riemann equations is given on page 85.) Anyone who uses the equations regularly should know it by heart.

As another example, outside of teaching I rarely use half angle formulae (or equivalently double angle formulae), for example, $\cos t = 1 - 2\sin^2(t/2)$. However, I can deduce them in seconds from the sine and cosine addition formulae (which I do know). I know this will horrify some of my colleagues as they know them by heart but as I rarely use them I feel that I don't need to keep them in my head. It leaves space for other information!

## How to remember the sine formula

In case you are interested, here's a method to remember the sine formula that was mentioned in the previous section. In essence, it is the method from Exercise 1.14(xviii):

We have two simple facts that we should certainly know without looking them up as they are used so often: (1) $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$, and (2) $e^{i(A+B)} = e^{iA}e^{iB}$ for $A, B \in \mathbb{R}$. Thus we have

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ \cos(A+B) + i\sin(A+B) &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B - \sin A\sin B \\ &\qquad + i(\sin A\cos B + \cos A\sin B). \end{aligned}$$

Equating real and imaginary parts gives *both* the sine and cosine addition formulae.

Sometimes, when a student half-remembers the formula but is unsure whether it is plus or minus or whether it involves sine times cosine or sine times sine, then I ask them to write their guess and then use $B = 0$ and $B = \pi$ to check it. In most guesses, doing that the correct formula becomes clear.

## Expanding power series and calculating residues

The preceding sections are rather abstract and so I will finish on an example of a specific technique that can save time. The broader abstract principle to learn is that one should always look for places where calculations can be simplified. Lecturers often do not give the most efficient way of doing something. There are various reasons for this approach, the most efficient way may lack any motivation or it may involve learning too much in one go. Hence it is often useful to return to a topic and ask how the calculations could be simplified.

Method D for calculating residues in Chapter 17 can involve some tiresome calculations of Laurent series. One way to avoid tedium is to note that we only need the coefficient of $z^{-1}$ and hence we do not need to calculate the whole series.

For example, consider $\dfrac{\sin z}{z^6(2z-1)}$ at 0. The $z^6$ in the denominator means that we should find the coefficient of $z^5$ in the Taylor series expansion of $\dfrac{\sin z}{2z-1}$. (Note that it is Taylor series rather than Laurent as this new function is defined and differentiable at $z = 0$.) We can find the coefficient of $z^5$ without calculating the whole Taylor series (or at least not all the terms up to $z^5$). To see how do we do this let's first see what we have to do to calculate the Taylor series.

We can multiply the Taylor series of $\sin z$ with that of $1/(2z-1)$. These series are

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + \dots, \text{ and}$$
$$\frac{1}{2z-1} = -\frac{1}{1-2z} = -\sum_{n=0}^{\infty}(2z)^n = -1 - 2z - 4z^2 - 8z^3 - 16z^4 - 32z^5 + \dots$$

Multiplying these out is complicated, time-consuming and it is easy to make a small error. However, we need only the $z^5$ terms and so require the sum of the coefficients arising from multiplying the constant and $z^5$ terms in the respective series, the $z$ and $z^4$ terms and so on.

This is easily done by drawing a quick table:

| | | $z^0$ | $z^1$ | $z^2$ | $z^3$ | $z^4$ | $z^5$ |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 0 | $-\frac{1}{6}$ | 0 | $\frac{1}{120}$ |
| $z^0$ | $-1$ | | | | | | $-\frac{1}{120}$ |
| $z^1$ | $-2$ | | | | | 0 | |
| $z^2$ | $-4$ | | | | $\frac{2}{3}$ | | |
| $z^3$ | $-8$ | | | 0 | | | |
| $z^4$ | $-16$ | | $-16$ | | | | |
| $z^5$ | $-32$ | 0 | | | | | |

The $z^5$ coefficient is the sum of the terms on the (anti-)diagonal, in this case

$$-16 + \frac{2}{3} - \frac{1}{120} = -\frac{1841}{120}.$$

Hence, we can now find the residue we seek:

$$\operatorname{res}\left(\frac{\sin z}{z^6(2z-1)}, 0\right) = -\frac{1841}{120}.$$

Once this method is understood one can streamline it further by not drawing a table but just writing down the relevant products.