CHAPTER 22

Common Mistakes

In this chapter we look at common errors made by students learning Complex Analysis. As a lecturer with many years of experience of teaching the subject I have seen these examples appear again and again in examinations. I'm sure that, due to pressure, we've all written nonsense in an exam which under normal conditions we wouldn't have. Nonetheless, many of these errors occur every year and I suspect something deeper is going on.

What follows is not intended to be a criticism of my students, who, luckily for me, are generally hard-working and intelligent. Nor is it an attempt to mock or ridicule them. Instead the aim is to identify common mistakes so that they are not made in the future.

And if this chapter seems negative in tone, the next is more positive as it delves into techniques that improve understanding.

Imaginary numbers cannot be compared

The first mistake is the probably the most common: the comparison of imaginary numbers. For example, students write $z < R$ for z a complex number. This cannot be right. If z were $3 + 2i$ what does it mean for $3 + 2i$ to be less than R? What is usually intended is the *modulus* of z, i.e., $|z| < R$.

The point is, unlike real numbers, we cannot order the complex numbers. For example, which is bigger $3+2i$ or $1+4i$? This is difficult to decide! Since complex numbers can be identified with the plane ordering them is equivalent to ordering the points of the plane and clearly this can't be done – at least not in any useful or meaningful way. One last point needs to be made. Although $z < 3 + 2i$ is *incorrect*, note that expressions like $z < 4$ can be true if z is a real number.

Not realizing re^{it} gives a circle

This may not count as a mistake but is such a common gap in student exam knowledge that it needs to be mentioned: Too many students can't sketch the image of e^{it} and/or can't write down a contour whose image is the circle.

Let's reiterate the basics. The contour defined by $\gamma(t) = re^{it}$ defines part of a circle of radius r. A full circle can be given by $0 \le t \le 2\pi$. Furthermore, the circle can be centred at w just by adding w. That is, $\gamma(t) = w + re^{it}$ with $0 \le t \le 2\pi$ produces a circle of radius r based at w.

By taking the modulus, we can show that $\gamma(t) = re^{it}$ really does give a circle:

$$\left|re^{it}\right| = r\left|\cos t + i\sin t\right| = r\left(\sqrt{\cos^2 t + \sin^2 t}\right) = r\sqrt{1} = r.$$

Thus, the points on the contour all have the same length, i.e., are the same distance from the origin. And, a circle is just all the points the same distance from a specified point. Therefore, the image of re^{it} gives a circle when $0 \le t \le 2\pi$.

$|e^z|$ is not equal to $e^{|z|}$

Tempting as it may be to believe, $|e^z|$ is not equal to $e^{|z|}$. These two expressions are related as we shall see. The correct equality is

$$|e^z| = e^{\mathrm{Re}(z)}.$$

It is reasonable to know the derivation of this:

$$|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = |e^x| \times 1 = |e^x| = e^x = e^{\mathrm{Re}(z)}.$$

What is the relation to $e^{|z|}$? It's an inequality:

$$|e^z| \le e^{|z|}.$$

This follows from the equality $|e^z| = e^{\mathrm{Re}(z)}$ since exp is a strictly increasing function and $\mathrm{Re}(z) \le |z|$, as we can see by drawing an Argand diagram. This last part is just Pythagoras' Theorem in action!

Limits of the Standard Geometric Series

Some students have trouble remembering whether the lower limit of the standard geometric series

$$\sum x^n = \frac{1}{1-x}$$

is $n = 0$ or $n = 1$.

Here is a situation in which taking a simple case provides the answer. We don't need to look it up or provide a proof. Let $x = 0$, then we have

$$\sum_{n=0}^{\infty} x^n = 1 + 0 + 0^2 + 0^3 + \cdots = \frac{1}{1-0}$$

and

$$\sum_{n=1}^{\infty} x^n = 0 + 0^2 + 0^3 + \cdots \neq \frac{1}{1-0}$$

So the lower limit is $n = 0$.

This exemplifies a useful technique: when unsure of recalling a result try a special case.

Basic definitions

It is possible to write a whole chapter on definitions and the problems that arise. (In fact, I did. See my book *How to Think Like a Mathematician.*)

Central definitions in complex analysis include differentiation and contour integral. I regularly ask for the definition of complex differentiation and regularly most of the students fail to state it correctly. Quite often what I get is

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}. \tag{22.1}$$

Alternatives include $f(z) \to f(a)$ as $z \to a$ or some garbled version of the above, for example with modulus signs, $z + h$ in the denominator and so on.

So what is wrong with the expression in (22.1)? For a start there is no explanation of what f, z and h are. It is important that f is a complex function and that z is identified as the point at which we are defining differentiability. Thus, we should say something like 'The complex function f is differentiable at z if . . . '. Then we should bring in (22.1), making clear that we want this limit to exist, and to exist when $h \in \mathbb{C}$ rather than $h \in \mathbb{R}$.

Thus the definition should be something like 'The complex function f is differentiable at z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}, \quad h\in\mathbb{C},$$

exists'.

Similar problems occur for the definition of contour integration. That is, what γ and f are in $\int_\gamma f(z)\,dz$ is unexplained.

This may seem pedantic. It is. Pedantry is very important in mathematics.

Confusing definitions with calculation processes

Do not confuse the *definition* of an object with a *process* by which we find that object.

A good example of this mistake was mentioned in Common Error 16.20. Poles and their multiplicities are clearly central to working with residues and so I regularly ask for the definitions in exams.

In response to 'Define what it means to be a pole at w and define the order of a pole' I receive inaccurate (and often long and rambling) descriptions of how a pole is found in certain situations. For example, 'The order of the pole is the power of the thing when the pole is zero' or 'the order of the pole is the order of the bracket'. (These get no marks.) You can see in the former that the student does have some partial knowledge of pole order but that it is dependent on how pole order is calculated, i.e., we look for the zero (in the denominator) and find in most cases the power to which $(z-w)$ is taken. However, the question asked for the definition, so the definition (in this case Definition 16.16) should be given.

As another example, consider the definition of a residue. Since the most useful method for calculating a residue is Method B, I often receive $\text{res}(p/q,w) = p(w)/q'(w)$ instead of a statement involving the coefficient of $(z-w)^{-1}$. Again, the student is seeing 'residue' as something that is calculated and gives a calculation method rather than seeing 'residue' as a concept.

Misstating Theorems - Insufficient detail

Following on from the previous example, if I had asked students to state Method B, then the above answer, $\text{res}(p/q,w) = p(w)/q'(w)$ would be insufficient. There is no explanation of what p, q are and what conditions are placed on them.

This is a common problem. When I ask students in my geometry class or during talks I give in schools what Pythagoras' Theorem is, I receive a prompt reply: $c^2 = a^2 + b^2$. I usually, to the initial confusion of the students, say no. There are two points, one is that a, b and c are not defined. The second is deeper. The equation $c^2 = a^2 + b^2$ is the *conclusion* of the theorem. The *assumptions* are missing. The most crucial of which of course is that we need a right-angled triangle. Students certainly know this detail but ignore its importance.

Hence, for Method B we need to say 'We have $\operatorname{res}(p/q, w) = p(w)/q'(w)$ for p, q analytic with $p(w) \neq 0$, $q(w) = 0$, $q'(w) \neq 0$'.

The most common problem in misstating theorems is to just state the conclusion, particularly if the conclusion is a formula. Hopefully, now you can see that Cauchy's Integral Formula is not just

$$\int_\gamma \frac{f(z)}{z-w}\,dz = 2\pi i\, n(\gamma, w) f(w).$$

Order of Poles

Another error with poles is to believe that the order of a pole is the order of the zero in the denominator of a rational function. That this is erroneous we can see in Example 16.17(iv). The function $(e^z - 1)/z^3$ has a zero of order 3 at 0 in the denominator but the order of the pole there is 2. In non-rigorous terms – and this really is non-rigorous – the numerator has a zero of order 1 and it 'cancels' with one of the zeros in the denominator.

We can't even say that the order of the pole is at most the order of the zero in the denominator. From Example 15.6(ii) we can deduce that $\cot(z)/z^2$ has a pole of order 3 at 0, not a pole of order 2. In this case the non-rigorous explanation is that cot has a pole of order 1 at 0 so combines with the pole of order 2 for $1/z^2$ to give a pole of order 3.

This partly arises from the non-uniqueness of the representation of a function as a quotient. Here $\cot z$ is $1/\tan z$ and so $\cot(z)/z^2$ can be written as $1/(z^2 \tan(z))$. The latter representation has a zero of order 3 in the denominator (since $\tan z$ has a zero of order 1 at 0) and is non-zero in the numerator, hence we have a pole of order 3. The point, perhaps, is to not be fooled by the way a function is written as a quotient.

Integrals

Clearly, an integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$, where f is a real function, must produce a real number. The methods in this book allow us to calculate such integrals with complex analysis and there is the danger that a minor miscalculation will produce an imaginary number. Hence, any working which produces an imaginary number is wrong and should be corrected.

The radius of convergence is real

The radius of convergence is real and *never* has an imaginary part.

The ratio test is one of the best tests we have for convergence of series and it can be used to calculate the radius of convergence of power series. In most elementary analysis course where real series $\sum_{n=0}^{\infty} a_n z^n$ are studied it is common for the ratio test to be stated with $a_n > 0$. This leads to some students mis-applying it in the complex case. Consider the series $\sum_{n=0}^{\infty}(3-4i)^n z^n$ and let $a_n = (3-4i)z^n$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{(3-4i)^{n+1}z^{n+1}}{(3-4i)^n z^n} = (3-4i)z \to (3-4i)z \text{ as } n \to \infty.$$

So far, so good. For this example, the common mistake is to write something like, 'we require that $z < \dfrac{1}{3-4i}$ and so the radius of convergence is $\dfrac{1}{3-4i}$'. (My guess is that students are slavishly following the procedure in the real case.) This cannot be right. What does a circle of radius $1/(3-4i)$ look like? We need the radius to be real.

To prevent this error we could define $a_n = |(3-4i)z^n|$ or, what amounts to the same thing, just write

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3-4i)^{n+1}z^{n+1}}{(3-4i)^n z^n}\right| = |(3-4i)|\,|z| = 5|z|.$$

Thus we require $5|z| < 1$, i.e., $|z| < 1/5$. In other words the radius of convergence is 1/5.

$\cos z$ is not $(z+z^{-1})/2$

I often see $\dfrac{z+z^{-1}}{2}$ substituted for $\cos z$ (and a similar substitution for $\sin z$) However, these are not equal. The confusion here comes from Chapter 20 where $\dfrac{z+z^{-1}}{2}$ is legitimately substituted for $\cos\theta$. The important differences are (i) θ is used, and (ii) it can only be used because $z = e^{i\theta}$.

Argument problems

The argument of a complex number causes a number of problems.

First, for the argument θ of $x+iy$ with $x \neq 0$ we have the equation $\tan\theta = y/x$. This means many students calculate θ with $\arctan(y/x)$ (sometimes also written, somewhat erroneously, as $\tan^{-1}(y/x)$). We can see that this can lead to errors:

$$\operatorname{Arg}(1+i) = \frac{\pi}{4} \quad \text{and} \quad \operatorname{Arg}(-1-i) = -\frac{3\pi}{4}$$

but $\arctan(1/1) = \arctan(1) = \arctan(-1/-1)$. Further details can be found in Common Error 1.7. The important point is to take care with the cases $x < 0$ and $x = 0$. Plotting on an Argand diagram often helps.

A second mistake is in the use of polar notation: If $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$, then $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$. The mistake is to forget the extra $2k\pi$ (and sometimes students take $k \in \mathbb{N}$ rather than $\mathbb{Z}$).

This leads to another common error as the previous remark has an important consequence for solving $e^z = w$. We have $z = \ln|w| + i\arg(w) + 2k\pi i$, for $k \in \mathbb{Z}$. Too often the $2k\pi i$ term is forgotten.

Odds and ends

(i) The modulus of a number is never complex. (This usually occurs due to erroneously taking $|x+iy| = \sqrt{x^2+(iy)^2}$! This is a very, very common mistake. It leads to $\sqrt{x^2-y^2}$ which can in turn lead to an imaginary number. See Common Error 1.6.)

(ii) $f(a+ib)$ is **not** equal to $f(a)+if(b)$. This error occurs more often than I find comfortable.

(iii) If z is complex, then $z \to \infty$ has not been defined. However, $|z| \to \infty$ is ok.

(iv) If $f(x+iy) = u(x,y) + iv(x,y)$, then v_x is **not** $\frac{\partial}{\partial x}(iv)$.

(v) The definition of contour integral does not include a modulus.

(vi) Cauchy's Theorem requires that the path γ is *closed*.

Summary

- Complex numbers cannot be compared with < and > signs.
- $w + re^{it}$ gives part or all of a circle.
- $|e^z| \leq e^{|z|}$ and is not an equality in general but $|e^z| = e^{\mathrm{Re}(z)}$.
- The lower limit of $\sum x^n = \dfrac{1}{1-x}$ is $n = 0$ not $n = 1$.
- Reproduce basic definitions verbatim.
- Don't confuse definitions with calculation processes.
- State theorems correctly – include the assumption, don't just give the conclusion, particularly for formulae.
- The order of a pole is not 'the order of the zero in the denominator'.
- Real integrals do not evaluate to give a complex number.
- The radius of convergence is real, it **never** has an imaginary part. Take the modulus when working out $\dfrac{a_{n+1}}{a_n}$ for radius of convergence.
- $\cos z$ is **not** equal to $\dfrac{z + \frac{1}{z}}{2}$. Similarly for sin.
- The argument of a complex number can't always be calculated by just $\tan^{-1}(y/x)$.
- Don't forget that the angle θ and $\theta + 2k\pi$ are the same for $k \in \mathbb{Z}$ and so in many formulae there is an extra term to deal with this ambiguity.