CHAPTER 11

Cauchy's Theorem

We now come to the fundamental theorem in complex analysis. There is no analogue in real analysis and it has far reaching, deep and surprising consequences, including applications to real variable problems. The rest of the book relies on this theorem.

A proof of the theorem is usually the most difficult part of an introductory complex analysis course. The most general proofs are often complicated, long, and do not give any insight into why the result is true. In this chapter we will state the theorem in generality and give a reasonably simple proof for a case that will be sufficient for all the later results in the book. A full proof is given in Appendix A.

Theorem 11.1 (Cauchy's Theorem)
Let $D \subseteq \mathbb{C}$ be an open set, and $f : D \to \mathbb{C}$ be a differentiable complex function. Let γ be a closed contour such that γ and its interior points are in D.

Then, $\displaystyle\int_\gamma f = 0$.

Remarks 11.2

(i) This is truly a great theorem. It refers to *any* open set in $\mathbb{C}$, *any* differentiable function on D, and *any* contour with all interior points in D. And it says that *any* integral arising from this is zero. Thus, weak assumptions lead to a strong conclusion.

(ii) It is necessary that the interior of γ lies within D. Let $D = \mathbb{C}\backslash\{w\}$ for any point in $w \in \mathbb{C}$. Suppose that $f(z) = 1/(z - w)$ and that γ is *any* contour that has w in its interior. Then, f is differentiable on D and

$$\int_\gamma f = \int_\gamma \frac{dz}{z-w} = 2\pi i\, n(\gamma, w) \neq 0.$$

(iii) Note that we can't just use the Fundamental Theorem of Calculus as in Corollary 8.11 since we don't know whether f has an antiderivative on D. (And usually it won't have.)

(iv) At first sight it may appear that the theorem will only tell us about the behaviour of differentiable functions. However, it has strong implications for non-differentiable functions as well as we will see later in Cauchy's Residue Theorem.

Exercise 11.3
Using the contours in Exercises 2 Question 1, to which of the following integrals does Cauchy's theorem apply? (There is no need to evaluate them.)

$$\int_{\gamma_1} |z|^2\, dz, \qquad \int_{\gamma_1} \frac{z^2}{z-2}\, dz, \qquad \int_{\gamma_2} z\, dz, \qquad \int_{\gamma_2+\gamma_3} \sin\left(\frac{1}{z-1}\right) dz.$$

Proof of a simple version of Cauchy's Theorem

We shall now begin to prove a weaker version of the theorem.

Definition 11.4
A **square** is a set of the form

$$\{z \in \mathbb{C} \mid a \leq \mathrm{Re}(z) \leq b, c \leq \mathrm{Im}(z) \leq d \text{ for some } a, b, c, d \in \mathbb{R} \text{ with } b - a = d - c > 0\}.$$

Lemma 11.5 (Nested Squares Lemma)
Suppose that $Q_1 \supset Q_2 \supset Q_3 \supset \ldots$ is a sequence of squares, then $\bigcap_{n=1}^{\infty} Q_n \neq \emptyset$.

Proof. Let $Q_n = \{z \in \mathbb{C} \mid a_n \leq \mathrm{Re}(z) \leq b_n,\, c_n \leq \mathrm{Im}(z) \leq d_n\}$. By assumption $[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \ldots$ and so we have

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n < \cdots < b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$$

for all n. Thus (a_n) is a bounded increasing sequence and hence converges to some a_0 with $a_n \leq a_0$. Similarly the sequence b_n has a limit b_0 with $b_0 \leq b_n$. As

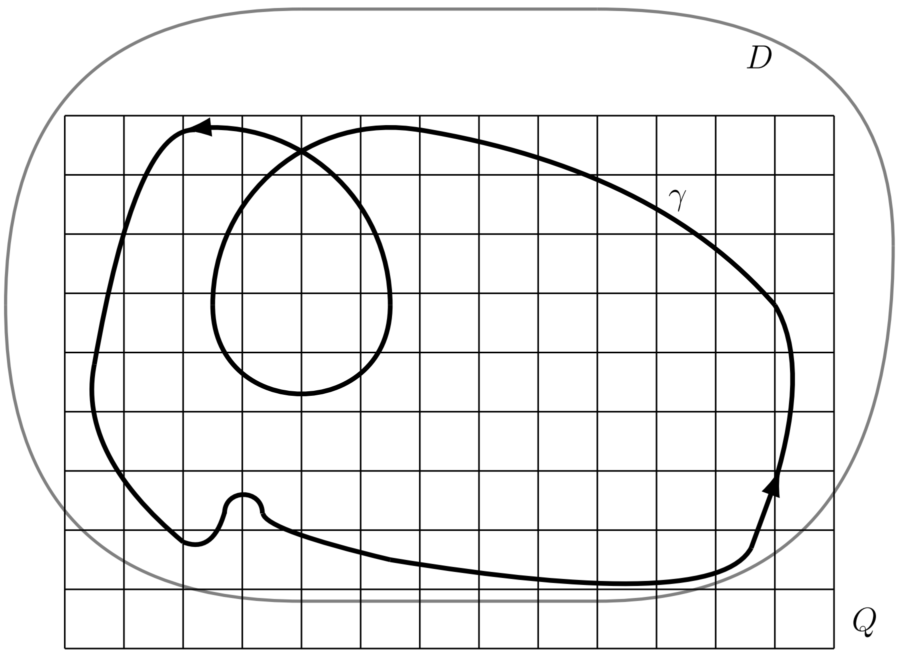
$a_n < b_n$ we have $a_0 \leq b_0$. Now $a_0 \in [a_n, b_n]$ for all n as $a_n \leq a_0 \leq b_0 \leq b_n$. So $a_0 \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. Similarly there exists $c_0 \in \bigcap_{n=1}^{\infty} [c_n, d_n]$ for some c_n, d_n.

Therefore $a_0 + ic_0 \in \bigcap_{i=1}^{\infty} Q_n$. □

Lemma 11.6
Let γ be a closed contour made of a finite number of lines and arcs in the open set D with $\widetilde{D} = \gamma^* \cup \text{Int}(\gamma) \subseteq D$. Let Q be a square in $\mathbb{C}$ bounding $\widetilde{D}$ and $f : D \to \mathbb{C}$ be analytic. Then for any $\epsilon > 0$ there exists a subdivision of Q into a grid of squares so that for each square Q_j in the grid with $Q_j \cap \widetilde{D} \neq \emptyset$ there exists a $z_j \in Q_j \cap \widetilde{D}$ such that

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon \quad \text{for all } z \in Q_j \cap \widetilde{D}.$$

Proof. The set up looks like the following diagram.



For a contradiction assume that the statement is not true. Let $Q_1 = Q$ and divide Q_1 into 4 equal-sized squares. At least one of these squares will not satisfy the required condition in the lemma. Let Q_2 be such a square. Repeat the process to produce an infinite sequence of squares with $Q_1 \supset Q_2 \supset Q_3 \supset \ldots$. By the Nested Squares Lemma there exists $z_j \in \bigcap_{n=1}^{\infty} Q_n$.

As f is differentiable there exists $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon$$

for $|z - z_j| < \delta$. But as the size of the squares becomes arbitrarily small there must exist Q_N such that Q_N is contained in the disk $|z - z_j| < \delta$. This is a contradiction. □

Remark 11.7
From this we can deduce that we can subdivide the square Q by squares with side length $S/2^N$ where S is the length of the side of Q and N is a natural number.

Theorem 11.8 (A Simple Version of Cauchy's Theorem)
Let $D \subseteq \mathbb{C}$ be a open set and $f : D \to \mathbb{C}$ be a differentiable function. Let γ be a simple closed contour made of a finite number of lines and arcs such that $\gamma^* \cup Int(\gamma) \subset D$. Then

$$\int_\gamma f(z)dz = 0.$$

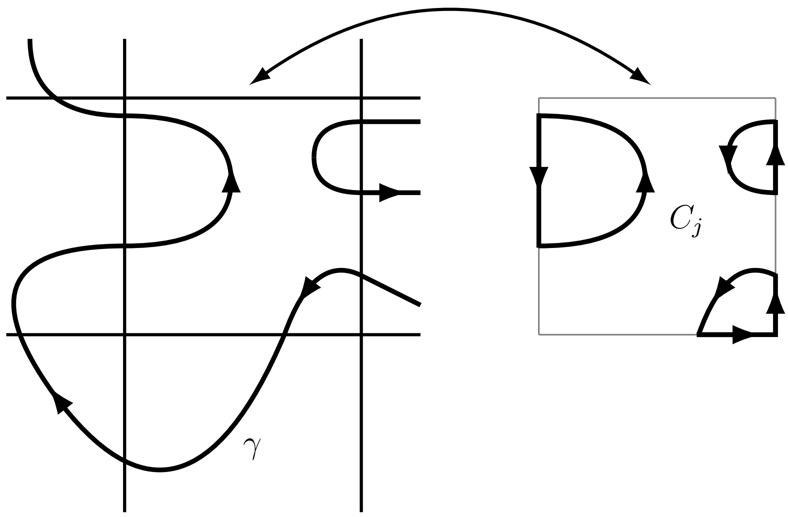
Proof. Given $\epsilon > 0$ there exists a grid of squares covering $\gamma^* \cup \text{Int}(\gamma)$ as in Lemma 11.6. Let $\{S_j\}_{j=1}^n$ be the set of squares such that $S_j \cap (\gamma^* \cup \text{Int}(\gamma)) \neq \emptyset$ and let $\{z_j\}_{j=1}^n$ be the set of distinguished points in the lemma.

Define $g_j : D \to \mathbb{C}$ by

$$g_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j), & z \neq z_j \\ 0, & z = z_j \end{cases}$$

Then as f is differentiable, g_j is continuous (and hence integrable).

Without loss of generality we can assume that γ is positively oriented. Let C_j be the union of positively oriented contours giving the boundary of $S_j \cap (\gamma^* \cup \text{Int}(\gamma))$. Since γ is made of a finite number of lines and arcs C_j will itself be the union of a finite number of lines and arcs. For S_j such that $S_j \cap \gamma^* = \emptyset$, C_j^* is just the boundary of a square. An example of a C_j consisting of three disjoint contours can be seen in the following diagram.



On S_j we have

$$f(z) = f(z_j) + (z - z_j)f'(z_j) + (z - z_j)g_j(z). \tag{11.1}$$

As $f(z_j) + (z - z_j)f'(z_j)$ is the derivative of

$$(z - z_j)f(z_j) + \frac{(z - z_j)^2}{2}f'(z_j)$$

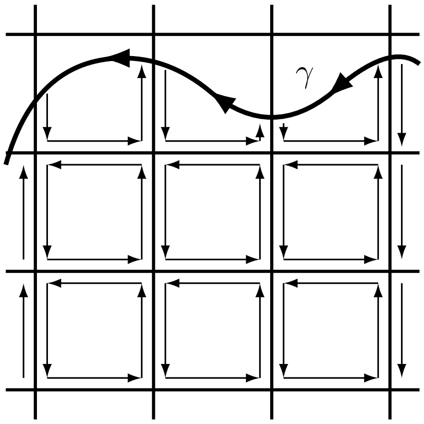
by the Fundamental Theorem of Calculus and the fact that C_j is closed we get

$$\int_{C_j} f(z_j) + (z - z_j)f'(z_j)dz = 0. \tag{11.2}$$

Now,

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz$$

and edges of touching squares will cancel. This is pictured in the following diagram.



So

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\sum_{j=1}^{n}\int_{C_j} f(z)dz\right| \\ &\le \sum_{j=1}^{n}\left|\int_{C_j} f(z)dz\right| \\ &= \sum_{j=1}^{n}\left|\int_{C_j} (z-z_j)g_j(z)dz\right| \qquad \text{by (11.1) and (11.2).}\end{aligned}$$

We now estimate each of the integrals in the sum.

Let s be the length of the side of the squares. For $z, z_j \in S_j$ we have

$$|(z-z_j)g_j(z)| < \sqrt{2}s\epsilon$$

because $|z - z_j| \le \sqrt{2}s$ as S_j is a square and $|g_j(z)| < \epsilon$ as the grid of squares satisfies the conclusion of the lemma.

Let l_j be the length of the curve(s) in $S_j \cap \gamma^*$ (the length may be zero). Then

$$L(C_j) \le l_j + 4s.$$

Hence, by the Estimation Lemma

$$\left|\int_{C_j} (z-z_j)g_j(z)dz\right| < \sqrt{2}s\epsilon(l_j + 4s).$$

Therefore

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &< \sum_{j=1}^{n} \sqrt{2}s\epsilon(l_j + 4s) \\ &= \sqrt{2}\epsilon \sum_{j=1}^{n}(sl_j + 4s^2) \\ &= \sqrt{2}\epsilon(sL(\gamma) + 4A)\end{aligned}$$

where A is the area of all the squares $\{S_j\}_{j=1}^n$. Now s is less than or equal to the length S of the side of the original square enclosing D. Hence,

$$\left|\int_\gamma f(z)dz\right| < \sqrt{2}\epsilon(SL(\gamma) + 4S^2).$$

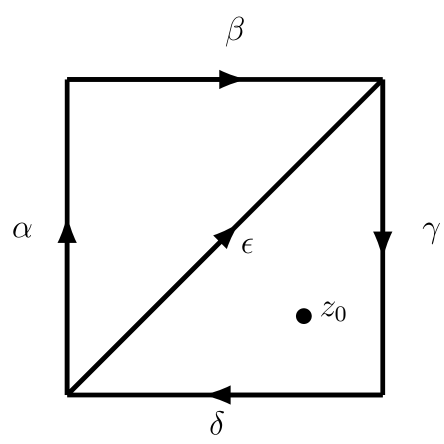
As ϵ was arbitrary and S and $L(\gamma)$ are fixed we have

$$\int_\gamma f(z)dz = 0.$$

□

Remark 11.9
The reason for having a finite number of lines and arcs is because then we know that each C_j^* consists of a finite number of disjoint contours. If we allowed any closed contour, then in principle it is possible to get an infinite number of pieces in a C_j^*. Hence we would have to worry about an infinite sum. This problem is not impossible to overcome but at the moment we wish to have a version of the theorem good enough for our proofs and applications.

Example 11.10
Consider the following diagram of contours.



Let f be a function analytic on and within the square except at the point z_0. We can show that

$$\int_{\alpha+\beta+\gamma+\delta} f\,dz = \int_{\gamma+\delta+\epsilon} f\,dz$$

as follows.

As f is complex differentiable $\int_{\alpha+\beta-\epsilon} f = 0$ and so

$$\int_{\alpha+\beta+\gamma+\delta} f = \int_{\alpha+\beta-\epsilon+\epsilon+\gamma+\delta} f = \int_{\alpha+\beta-\epsilon} f + \int_{\epsilon+\gamma+\delta} f = \int_{\gamma+\delta+\epsilon} f.$$

In theory the integrals of a function over two separate contours could be distinct. The following proposition shows that, in certain circumstances, the integrals are independent of the path taken.

Proposition 11.11
Suppose that α and β are contours such that their start points coincide and their end points coincide. Further suppose that $f : D \to \mathbb{C}$ is a function analytic on an open set D such that $\alpha - \beta$ and the interior of $\alpha - \beta$ are in D. Then,

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

Proof. The result follows from

$$\int_\alpha f(z)\,dz - \int_\beta f(z)\,dz = \int_{\alpha-\beta} f(z)\,dz = 0.$$

□

Exercises

Exercises 11.12

(i) Let

$$\begin{aligned} D_1 &= \mathbb{C}, \\ D_2 &= \{z \in \mathbb{C} \,|\, |z| \le 4\}, \\ D_3 &= \{z \in \mathbb{C} \,|\, |z| < 4\}, \\ D_4 &= \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \ge -1\}, \\ D_5 &= \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \ge -10\}, \\ f_1(z) &= z^2 + 3z - 5, \\ f_2(z) &= \frac{1}{z^2+4}, \\ f_3(z) &= \frac{1}{z+2i}, \\ C_r(t) &= re^{it}, 0 \le t \le 2\pi, \\ \Gamma &= \text{boundary of } \{z \in \mathbb{C} \,|\, |z| \le 10,\ \mathrm{Im}(z) \ge 0\}. \\ \beta(t) &= t, -1 \le t \le 1. \end{aligned}$$

Consider the following open sets, functions and contours. Which of the combination satisfy the assumptions of Cauchy's Theorem? Where the assumptions are not satisfied, then explicitly explain why.

(a) Let $D = D_2$, $f = f_1$, $\gamma = C_1$.

(b) Let $D = D_3$, $f = f_1$, $\gamma = C_1$.

(c) Let $D = D_3$, $f = f_1$, $\gamma = C_4$.

(d) Let $D = D_3\backslash\{-2i\}$, $f = f_2$, $\gamma = C_1$.

(e) Let $D = D_3\backslash\{-2i\}$, $f = f_3$, $\gamma = C_1$.

(f) Let $D = D_3\backslash\{-2i\}$, $f = f_3$, $\gamma = C_3$.

(g) Let $D = D_1$, $f = f_1$, $\gamma = \beta$.

(h) Let $D = D_1$, $f = f_1$, $\gamma = \beta + C_1 - \beta$.

(i) Let $D = D_5$, $f = f_2$, $\gamma = \Gamma$.

(j) Let $D = D_4\backslash\{2i\}$, $f = f_2$, $\gamma = \Gamma$.

(k) Let $D = D_4$, $f = f_3$, $\gamma = \Gamma$.

(l) Let $D = D_5$, $f = f_3$, $\gamma = \Gamma$.

(ii) (Existence of antiderivatives II.) Let U be an open disc in $\mathbb{C}$ centred at w and $f : U \to \mathbb{C}$ be a differentiable function on U. Let $\Gamma(z)$ be the straight line contour from w to z for each $z \in U$. That is, $\Gamma(z)(t) = (t-1)w + tz$ for all $0 \le t \le 1$.

Show that $F(z) = \int_{\Gamma(z)} f(\zeta)\, d\zeta$ is an antiderivative of f on U, i.e., $F'(z) = f(z)$ for all $z \in U$.

Why is this not a counterexample to Remark 11.2(iii)?

(iii) Let D be a path connected open set in $\mathbb{C}$ and $f : D \to \mathbb{C}$ be a differentiable function. Let γ_1 and γ_2 be closed contours in D such that $n(\gamma_1, z) = -n(\gamma_2, z)$ for all $z \notin D$. Show that

$$\int_{\gamma_1} f(z)\, dz = -\int_{\gamma_2} f(z)\, dz.$$

[Hint: Consider a contour from a point of image of γ_1 to one of γ_2.]

(iv) Suppose that $f : D \to \mathbb{C}$ is a complex analytic function on the open set D such that f is complex differentiable except at $w \in D$. Prove there exists $R > 0$ such that for all $0 < r < R$ we have

$$\int_{C_R} f(z)\, dz = \int_{C_r} f(z)\, dz.$$

(This is an important exercise and a proof will be given later. Nonetheless, is worthwhile to work this out.)

(v) Suppose that γ_1 and γ_2 are two contours such that $\gamma_2^* \subset \mathrm{Int}(\gamma_1)$. Let f be a function that is differentiable on γ_1^* and γ_2^* and $\mathrm{Int}(\gamma_1)\backslash\mathrm{Int}(\gamma_2)$.

Prove that, if $n(\gamma_1, w) = n(\gamma_2, w)$ for all $w \in \mathrm{Int}(\gamma_2)$, then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

[Warning: Don't forget to carefully construct an open set upon which f is differentiable.]

(vi) Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where γ_1 is the contour given by going 0 to R on the real line, γ_2 goes anticlockwise in a circle arc to $e^{i\pi/4}R$ and γ_3 goes in a straight line from $e^{i\pi/4}R$ back to the origin. (This is called a **slice of pie contour** or **sector contour** due to its shape.)

(a) Show that $\int_\gamma e^{-z^2}\, dz = 0$.

(b) Show that $\int_{\gamma_2} e^{-z^2}\,dz \to 0$ as $R \to \infty$.

(c) Show that

$$\int_{\gamma_3} e^{-z^2}\,dz = -e^{i\pi/4}\int_0^R e^{-ix^2}\,dx.$$

(d) Hence, using the well-known equality $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, deduce the **Fresnel integrals**:

$$\int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{8}},$$
$$\int_0^\infty \cos(x^2)\,dx = \sqrt{\frac{\pi}{8}}.$$

(vii) If we add the assumption that f' is continuous to the statement of Cauchy's Theorem then the resulting theorem can be proved quite quickly using Green's Theorem. Either prove this or find a proof in a book or online.

(viii) Generalise Theorem 11.8 by dropping the assumption that the contour is simple.

Summary

❏ Let $D \subseteq \mathbb{C}$ be a open set, and $f : D \to \mathbb{C}$ be a differentiable complex function. Let γ be a closed contour such that γ and its interior points lie in D.

Then, $\displaystyle\int_\gamma f = 0$.